$\sum_{k=1}^{1} k^2 = 1^2 = 1 = \frac{2(2-1)(2\times 2-1)}{6}$ donc la proposition est vraie pour n = 2.

⊞ Soit p un entier ≥ 2. Supposons que $\sum_{k=1}^{p-1} k^2 = \frac{p(p-1)(2p-1)}{6}$

Montrons que $\sum_{k=1}^{p} k^2 = \frac{(p+1)p(2p+1)}{6}$. En effet

$$\sum_{k=1}^{p} k^2 = \sum_{k=1}^{p-1} k^2 + p^2 = \frac{p(p-1)(2p-1)}{6} + p^2$$
$$= \frac{p(p-1)(2p-1)+6p^2}{6} = \frac{p[(p-1)(2p-1)+6p]}{6}$$
$$= \frac{p[2p^2+3p-1]}{6} = \frac{p(p+1)(2p+1)}{6}$$

Conclusion: $\forall n \geq 2$; on a $\sum_{k=1}^{n-1} k^2 = \frac{n(n-1)(2n-1)}{6}$.

b- $S_n = \frac{1}{n}\sum_{k=1}^{n-1} f\left(\frac{k}{n}\right) = \frac{1}{n}\sum_{k=1}^{n-1}\left[\left(\frac{k}{n}\right)^2 - 2\text{Log}\left(\frac{k}{n}\right) - 1\right]$

$$= \frac{1}{n}\sum_{k=1}^{n-1}\left(\frac{k}{n}\right)^2 - 2\frac{1}{n}\sum_{k=1}^{n-1}\text{Log}\left(\frac{k}{n}\right) - \frac{1}{n}\sum_{k=1}^{n-1} 1$$

• $\frac{1}{n}\sum_{k=1}^{n-1}\left(\frac{k}{n}\right)^2 = \frac{1}{n^3}\sum_{k=1}^{n-1} k^2 = \frac{n(n-1)(2n-1)}{6n^3} = \frac{(n-1)(2n-1)}{6n^2}$.

• $\frac{-2}{n}\sum_{k=1}^{n-1}\text{Log}\left(\frac{k}{n}\right) = \frac{-2}{n}\text{Log}\left(\frac{1}{n}\times\frac{2}{n}\times\frac{3}{n}\times\ldots\times\frac{n-1}{n}\right)$

$$= \frac{-2}{n}\text{Log}\left(\frac{(n-1)!}{n^{n-1}}\right) = \frac{-2}{n}\text{Log}\left(\frac{n\times(n-1)!}{n\times n^{n-1}}\right)$$
$$= -\frac{2}{n}\text{Log}\left(\frac{n!}{n^n}\right) = \frac{2}{n}\text{Log}\frac{n^n}{n!} = 2\text{Log}\frac{n}{\sqrt[n]{n!}}.$$

• $\frac{1}{n}\sum_{k=1}^{n-1} 1 = \frac{1}{n}(n-1) = 1 - \frac{1}{n}$.

Ainsi $S_n = \frac{(n-1)(2n-1)}{6n^2} + 2\text{Log}\frac{n}{\sqrt[n]{n!}} - 1 + \frac{1}{n}$.

c- $\lim_{n\to+\infty} S_n = \lim_{n\to+\infty}\left[\frac{(n-1)(2n-1)}{6n^2} + 2\text{Log}\frac{n}{\sqrt[n]{n!}} - 1 + \frac{1}{n}\right]$.

* $\lim_{n\to+\infty} S_n = \frac{4}{3}$. * $\lim_{n\to+\infty}\frac{1}{n} = 0$

* $\lim_{n\to+\infty}\frac{(n-1)(2n-1)}{6n^2} = \lim_{n\to+\infty}\frac{n^2\left[2-\frac{3}{n}+\frac{1}{n^2}\right]}{6n^2} = \frac{1}{3}$.

D'où $\frac{4}{3} = \frac{1}{3} + 2\lim_{n\to+\infty}\text{Log}\frac{n}{\sqrt[n]{n!}} - 1 \Leftrightarrow \lim_{n\to+\infty}\text{Log}\frac{n}{\sqrt[n]{n!}} = 1$

$$\Leftrightarrow \lim_{n\to+\infty}\frac{n}{\sqrt[n]{n!}} = e.$$